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Highest weight irreducible representations of the quantum algebra $\mathbf{U}_h(\mathbf{A}_\infty)$

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Abstract. A class of highest weight irreducible representations of the algebra $U_h(A_\infty)$, the quantum analogue of the completion and central extension A_∞ of the Lie algebra gl_∞, is constructed. It is considerably larger than the known so far representations. Within each module a basis is introduced and the transformation relations of the basis under the action of the Chevalley generators are explicitly written. The verification of the quantum algebra relations to be satisfied is shown to reduce to a set of nontrivial q-number identities. All our representations are restricted in the terminology of S. Levendorskii and Y. Soibelman (Commun. Math. Phys. **140**, 399-414 (1991)).

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1. INTRODUCTION

We construct a class of highest weight irreducible representations (irreps) of the algebra $U_h(A_\infty)$ [8], the quantum analogue of the completion and central extension A_∞ of the Lie algebra gl_∞ [1], [6]. Our interest in the subject stems from the observation that certain representations of $gl(n)$ (including $n=\infty$) [9] and of A_∞ (see *Example 2* in Ref. [10] and the references therein) are related to a new quantum statistics, the A-statistics. The latter, as it is clear now, is a particular case of the Haldane exclusion statistics [5], a subject of considerable interest in condensed matter physics (see Sect. 4 in Ref. [13] for more discussions on the subject). It turns out that some of the representations of the deformed algebra $U_h(A_\infty)$ satisfy also the requirements of the Haldane statistics. More precisely, they lead to new solutions for the microscopic statistics of Karabali and Nair [7] directly in the case of infinitely many degrees of freedom. These results will be published elsewhere. We mention them here only in order to justify our personal motivation for the work we are going to present. One may expect certainly that similar as for A_∞ [2], [4] the representations of $U_h(A_\infty)$ may prove useful also in other branches of physics and mathematics.

The quantum analogues of gl_∞ and A_∞ in the sense of Drinfeld [3], namely $U_h(gl_\infty)$ and $U_h(A_\infty)$, were worked out by Levendorskii and Soibelman [8]. These authors have constructed a class of highest weight irreducible representations, writing down explicit expressions for the transformations of the basis under the action of the algebra generators.

The $U_h(A_\infty)$-modules, which we study, are labeled by all possible sequences (see the end of the introduction for the notation) $\{M\} \equiv \{M_i\}_{i\in\mathbf{Z}} \in \mathbf{C}[h]^\infty$, subject to the conditions:

(a) There exists $m \le n \in \mathbf{Z}$, such that $M_m = M_{m-k}$ and $M_n = M_{n+k}$ for all $k \in \mathbf{N}$;

(b) $M_i - M_j \in \mathbf{Z}_+$ for all $i < j \in \mathbf{Z}$.

Representations, corresponding to two different sequences, $\{M^1\} \neq \{M^2\}$, are inequivalent. The $U_h(A_\infty)$-modules of Levendorskii and Soibelman [8] consist of all those sequences $\{M^{(s)}\}$, for which $s = m = n \in \mathbf{Z}$ and $M_i^{(s)} = 1$, if $i < s$ and $M_i^{(s)} = 0$ for $i \ge s$.

In Refs. [11] and [12] a class of highest weight irreps, called finite signature representations, of the Lie algebra A_∞ was constructed. The corresponding modules are labeled by the set of all complex sequences $\{M\} \equiv \{M_i\}_{i\in\mathbf{Z}} \in \mathbf{C}^\infty$, which satisfy the conditions (a) and (b). The name "finite signature" comes to indicate that, due to (a), each signature $\{M\}$ is characterized by a finite number of different coordinates or, more precisely, by no more then $n-m+1$ different complex numbers.

From our results it follows that each $A_\infty-$module with a signature $\{M\}$ can be deformed to an $U_h(A_\infty)-$module with the same signature. The class of the finite signature representations of $U_h(A_\infty)$ is however larger, which is due to the fact that the coordinates of the $U_h(A_\infty)$ signatures take values in $\mathbf{C}[h]$. All representations we obtain are restricted in the sense of Ref. [8], Definition 4.1.

In Section 2 we construct a class of highest weight irreps of the subalgebra $U_h(a_\infty)$ of $U_h(A_\infty)$. Some of these representations, namely the finite signature representations (*Definition 2*), are extended to representations of $U_h(A_\infty)$ in Section 3.

Throughout the paper we use the notation (most of them standard):

N - all positive integers;

$\mathbf{Z}_+$ - all non-negative integers;

$\mathbf{Z}$ - all integers;

$\mathbf{Q}$ - all rational numbers;

$\mathbf{C}$ - all complex numbers;

$\mathbf{C}[h]$ - the ring of all polynomials in h over $\mathbf{C}$;

$\mathbf{C}[[h]]$ - the ring of all formal power series in h over $\mathbf{C}$;

$X^n = \{(a_1, a_2, \ldots, a_n)|\ a_i \in X\}$, (including $n = \infty$);

$[a, b] = \{x|\ a \le x \le b, x \in \mathbf{Z}\}, \quad (a, b) = \{x|\ a < x < b, x \in \mathbf{Z}\}$;

$\Gamma(\{M\})$ - the C-basis of a module with a signature $\{M\}$;

$\theta(i) = \begin{cases} 1, & \text{for } i \ge 0 \\ 0, & \text{for } i < 0; \end{cases}$

$q = e^{h/2} \in \mathbf{C}[[h]]$;

$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \in \mathbf{C}[[h]]$;

$[x, y]_q = xy - qyx$.

2. REPRESENTATIONS OF THE ALGEBRA $\mathbf{U}_h(a_\infty)$

The algebra $U_h(a_\infty)$ was defined in Ref. [8]. The authors denote it as $U_h(g'(A_\infty))_f$. It is a Hopf algebra, which is a topologically free module over $\mathbf{C}[[h]]$ (complete in h-adic topology), with generators $\{e_i, f_i, h_i, c\}_{i\in\mathbf{Z}}$, and

1. Cartan relations:

$$\begin{aligned}
&[c, a] = 0, \quad a \in \{h_i, e_i, f_i\}_{i\in\mathbf{Z}}\\
&[h_i, h_j] = 0,\\
&[h_i, e_j] = (\delta_{ij} - \delta_{i,j+1})e_j,\\
&[h_i, f_j] = -(\delta_{ij} - \delta_{i,j+1})f_j,\\
&[e_i, f_j] = \delta_{ij}\frac{q^{h_i - h_{i+1} + (\theta(-i) - \theta(-i-1))c} - q^{-h_i + h_{i+1} - (\theta(-i) - \theta(-i-1))c}}{q - q^{-1}}.
\end{aligned} \tag{1}$$

2. e-Serre relations:

$$\begin{aligned}
&e_i e_j = e_j e_i, \quad \text{if } |i - j| \neq 1,\\
&e_i^2 e_{i+1} - (q + q^{-1}) e_i e_{i+1} e_i + e_{i+1} e_i^2 = 0,\\
&e_{i+1}^2 e_i - (q + q^{-1}) e_{i+1} e_i e_{i+1} + e_i e_{i+1}^2 = 0.
\end{aligned} \tag{2}$$

3. f-Serre relations:

$$\begin{aligned}
&f_i f_j = f_j f_i, \quad \text{if } |i - j| \neq 1,\\
&f_i^2 f_{i+1} - (q + q^{-1}) f_i f_{i+1} f_i + f_{i+1} f_i^2 = 0,\\
&f_{i+1}^2 f_i - (q + q^{-1}) f_{i+1} f_i f_{i+1} + f_i f_{i+1}^2 = 0.
\end{aligned} \tag{3}$$

We do not write the other Hopf algebra maps ($\Delta,\ \varepsilon,\ S$) [8], since we will not use them. They are certainly also a part of the definition of $U_h(a_\infty)$.

Replacing throughout in the above relations $\{e_i, f_i, h_i, c\}_{i\in\mathbf{Z}}$ with $\{E_i, F_i, H_i, 0\}_{i\in\mathbf{Z}}$, one obtains the definition of $U_h(gl_\infty)$.

In terms of an equivalent set of generating elements $\{\hat{e}_i, \hat{f}_i, h_i, c\}_{i\in\mathbf{Z}}$, with

$$\hat{e}_i = e_i q^{(h_{i+1}-h_i)/2}, \quad \hat{f}_i = f_i q^{(h_i-h_{i+1})/2}, \tag{4}$$

one writes the quantum analogues of the Weyl generators $\{e_{ij}\}_{(i,j)\in\mathbf{Z}^2}$:

$$\begin{aligned} &e_{ii} = h_i, \quad e_{i,i+1} = \hat{e}_i, \quad e_{i+1,i} = \hat{f}_i, \\ &e_{ij} = [\hat{e}_i, [\hat{e}_{i+1}, [\ldots, [\hat{e}_{j-2}, \hat{e}_{j-1}]_q \ldots]_q]_q]_q, \quad i+1 < j, \\ &e_{ji} = [\hat{f}_i, [\hat{f}_{i+1}, [\ldots, [\hat{f}_{j-2}, \hat{f}_{j-1}]_q \ldots]_q]_q]_q, \quad i+1 < j. \end{aligned} \tag{5}$$

The "commutation relations" between these generators follow from (1)-(3) and are given in Ref. [8]. The relevance of the generators $\{e_{ij}\}_{(i,j)\in\mathbf{Z}^2}$ stems from the observation that the set of ordered monomials (see Ref. [8] for the ordering)

$$c^l \prod_{(i,j)\in\mathbf{Z}^2} e_{ij}^{n_{ij}} \tag{6}$$

with finitely many non-zero exponents $n_{ij} \in \mathbf{Z}_+,\ l \in \mathbf{Z}_+$ forms a (topological) basis in $U_h(a_\infty)$.

Set $H = \oplus_i \mathbf{C} h_i$, define linear functionals $\varepsilon_i : H \to \mathbf{C}$ by $\varepsilon_i(h_j) = \delta_{ij}$ and set $\alpha_i = \varepsilon_i - \varepsilon_{i+1},\ \ Q'_+ = \oplus_i \mathbf{Z}_+ \alpha_i$. Denote by $U_h(n_+)$ (respectively $U_h(n_-)$) the unital subalgebra in $U_h(a_\infty)$ generated by $\{e_i\}_{i\in\mathbf{Z}}$ (respectively $\{f_i\}_{i\in\mathbf{Z}}$). Then

$$U_h(n_\pm) = \oplus_{\alpha\in Q'_+} U_h(n_\pm)_{\pm\alpha}, \tag{7}$$

where

$$U_h(n_\pm)_{\pm\alpha} = \{x \in U_h(n_\pm) |\ [h', x] = \pm\alpha(h')x,\ \forall h' \in H\} \tag{8}$$

for $\alpha \neq 0$, and $U_h(n_\pm)_0 = \mathbf{C}[[h]]$. Any element $u \in U_h(a_\infty)$ can be represented as

$$u = \sum_{k=0}^{\infty} h^k \sum_{l=0}^{l(k)} c^l \sum_{\alpha,\beta\in Q'_+} \sum_{\gamma(k,l)\in\mathbf{Z}_+^\infty} \sum_{t=1}^{t(\alpha,\beta)} \mathcal{F}_{\alpha,k,l,t} \prod_{i\in\mathbf{Z}} h_i^{\gamma(k,l)_i} \mathcal{E}_{\beta,k,l,t}, \quad \textit{finite sums over } \alpha,\ \beta,\ \gamma, \tag{9}$$

where $\mathcal{F}_{\alpha,k,l,t} \in U_h(n_-)_{-\alpha},\ \mathcal{E}_{\beta,k,l,t} \in U_h(n_+)_{+\beta}$ and finitely many exponents $\gamma(k,l)_i$ are different from zero. The words "*finite sums over* $\alpha,\ \beta,\ \gamma$" have been added in order to indicate that for a fixed k only finitely many summands in (9) are different from zero.

The set $\hat{U}_h(a_\infty)$, consisting of all $\mathbf{C}[[h]]-$polynomials of the Chevalley generators $\{e_i, f_i, h_i, c\}_{i\in\mathbf{Z}}$, is dense in $U_h(a_\infty)$ with a basis (6). In particular $\mathcal{F}_{\alpha,k,l,t},\ \mathcal{E}_{\beta,k,l,t}$ and $\prod_{i\in\mathbf{Z}} h_i^{\gamma(k,l)_i}$ are in $\hat{U}_h(a_\infty)$. Then according to (9) any element $u \in U_h(a_\infty)$ is of the form

$$u = \sum_{k=0}^{\infty} u_k h^k, \qquad u_k = \sum_{l=0}^{l(k)} c^l \sum_{\alpha,\beta\in Q'_+} \sum_{\gamma(k,l)\in\mathbf{Z}_+^\infty} \sum_{t=1}^{t(\alpha,\beta)} \mathcal{F}_{\alpha,k,l,t} \prod_{i\in\mathbf{Z}} h_i^{\gamma(k,l)_i} \mathcal{E}_{\beta,k,l,t} \in \hat{U}_h(a_\infty). \tag{10}$$

We pass to construct a class of highest weight irreps of $U_h(a_\infty)$. To this end we define the $U_h(a_\infty)$-module $V(\{M\};\xi_0,\xi_1)$, labeled by $\xi_0,\ \xi_1 \in \mathbf{C}[h]$ and by a sequence $\{M\} \equiv \{M_i\}_{i\in\mathbf{Z}} \in \mathbf{C}[h]^\infty$ such that

$$M_i - M_j \in \mathbf{Z}_+,\ \ \forall i < j \in \mathbf{Z},$$

with a basis $\Gamma(\{M\})$. The latter, called a central basis (C-bases), is independent on ξ_0, ξ_1. It is formally the same as the one introduced in Refs. [11] and [12] for a description of representations of a_∞. $\Gamma(\{M\})$ consists of all $C-$ patterns

$$(M) \equiv \left[\begin{array}{cccccccc} .., & M_{1-\theta-k}, & \ldots, & M_{-1}, & M_0, & M_1, & \ldots, & M_{k-1}, ... \\ .., & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ & M_{1-\theta-k,2k+\theta-1}, & \ldots, & M_{-1,2k+\theta-1}, & M_{0,2k+\theta-1}, & M_{1,2k+\theta-1}, & \ldots, & M_{k-1,2k+\theta-1} \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & & M_{-1,3}, & M_{03}, & M_{13} & & \\ & & & M_{-1,2}, & M_{02} & & & \\ & & & & M_{01} & & & \end{array}\right], \quad (11)$$

where $k \in \mathbf{N},\ \theta = 0,1$. Each such pattern is an ordered collection of formal polynomials in h

$$M_{i,2k+\theta-1} \in \mathbf{C}[h], \quad \forall k \in \mathbf{N}, \quad \theta = 0,1, \qquad i \in [-\theta - k + 1, k-1], \tag{12}$$

which satisfy the conditions:

(i) there exists a positive, depending on (M), integer $N_{(M)} > 1$, such that

$$M_{i,2k+\theta-1} = M_i,\ \ \forall\ 2k+\theta-1 \geq N_{(M)}, \quad \theta = 0,1,\ \ i \in [1-\theta-k, k-1]; \tag{13a}$$

(ii) for each $k \in \mathbf{N},\ \theta = 0,1$ and $i \in [1-\theta-k, k-1]$

$$M_{i+\theta-1,2k+\theta} - M_{i,2k+\theta-1} \in \mathbf{Z}_+,\ M_{i,2k+\theta-1} - M_{i+\theta,2k+\theta} \in \mathbf{Z}_+. \tag{13b}$$

Denote by $\hat{V}(\{M\};\xi_0,\xi_1)$ the free $\mathbf{C}[[h]]$-module with generators $\Gamma(\{M\})$ and let $V(\{M\};\xi_0,\xi_1)$ be its completion in the h-adic topology. $V(\{M\};\xi_0,\xi_1)$ is a topologically free $\mathbf{C}[[h]]$-module with a (topological) basis $\Gamma(\{M\})$ and $\hat{V}(\{M\};\xi_0,\xi_1)$ is dense in it (in the h-adic topology). $V(\{M\};\xi_0,\xi_1)$ consists of all formal power series in h with coefficients in $\hat{V}(\{M\};\xi_0,\xi_1)$:

$$v = \sum_{i=0}^{\infty} v_i h^i, \quad v_0, v_1, v_2, \ldots \in \hat{V}(\{M\};\xi_0,\xi_1). \tag{14}$$

If a is a $\mathbf{C}[[h]]-$linear map in $\hat{V}(\{M\};\xi_0,\xi_1)$, $a \in End\ \hat{V}(\{M\};\xi_0,\xi_1)$, we extend it to a continuous linear map on $V(\{M\};\xi_0,\xi_1)$ setting

$$av = \sum_{i=0}^{\infty}(av_i)h^i. \tag{15}$$

Therefore the transformation of $V(\{M\};\xi_0,\xi_1)$ under the action of a is completely defined, if a is defined on $\Gamma(\{M\})$.

We proceed to turn $V(\{M\};\xi_0,\xi_1)$ into a $U_h(a_\infty)$ module. Denote by $(M)_{\pm\{j,p\}}$ and $(M)^{\pm\{l,q\}}_{\pm\{j,p\}}$ the patterns obtained from the C-pattern (M) in (11) after the replacements

$$M_{jp}\to M_{jp}\pm 1 \quad \text{and} \quad M_{lq}\to M_{lq}\pm 1, \quad M_{jp}\to M_{jp}\pm 1,$$

correspondingly, and let

$$S(j,l;\nu)=\begin{cases}(-1)^\nu & \text{for } j=l\\ \;\;1 & \text{for } j<l\\ -1 & \text{for } j>l\end{cases}, \quad \theta(i)=\begin{cases}1 & \text{for } i\geq 0\\ 0 & \text{for } i<0\end{cases}, \quad L_{ij}=M_{ij}-i. \tag{16}$$

Set moreover

$$e_i^0=f_i, \quad e_i^1=e_i, \quad i\in\mathbf{Z}. \tag{17}$$

Let $\{\rho(e_i),\rho(f_i),\rho(h_i),\rho(c)\}_{i\in\mathbf{Z}}$ be a collection of $\mathbf{C}[[h]]$-endomorphisms of $V(\{M\};\xi_0,\xi_1)$, defined on any C-pattern $(M)\in\Gamma(\{M\})$, as follows (see also (50), (51), (54) and (55)):

$$\rho(e_{-1}^{1-\mu})(M)=([L_{-1,2}-L_{0,1}-\mu][L_{0,1}-L_{0,2}+\mu])^{1/2}(M)_{-(-1)^\mu\{0,1\}}, \quad \mu=0,1\,, \tag{18}$$

$$\begin{aligned}\rho(e^{\mu}_{(-1)^\nu i-1})(M)=&-\sum_{j=1-i-\nu}^{i-1}\sum_{l=-i}^{i+\nu-1}S(j,l;\nu)\\ &\times\Bigg(-\frac{\prod_{k\neq l=-i}^{i+\nu-1}[L_{k,2i+\nu}-L_{j,2i+\nu-1}-(-1)^\nu\mu]\prod_{k=1-i}^{i+\nu-2}[L_{k,2i+\nu-2}-L_{j,2i+\nu-1}-(-1)^\nu\mu]}{\prod_{k\neq j=1-i-\nu}^{i-1}[L_{k,2i+\nu-1}-L_{j,2i+\nu-1}][L_{k,2i+\nu-1}-L_{j,2i+\nu-1}+(-1)^{\mu+\nu}]}\\ &\times\frac{\prod_{k=-i-\nu}^{i}[L_{k,2i+\nu+1}-L_{l,2i+\nu}+(-1)^\nu(1-\mu)]\prod_{k\neq j=1-i-\nu}^{i-1}[L_{k,2i+\nu-1}-L_{l,2i+\nu}+(-1)^\nu(1-\mu)]}{\prod_{k\neq l=-i}^{i+\nu-1}[L_{k,2i+\nu}-L_{l,2i+\nu}][L_{k,2i+\nu}-L_{l,2i+\nu}+(-1)^{\mu+\nu}]}\Bigg)^{1/2}\\ &\times(M)^{-(-1)^{\mu+\nu}\{l,2i+\nu\}}_{-(-1)^{\mu+\nu}\{j,2i-1+\nu\}}, \quad i\in\mathbf{N}, \quad \mu,\nu=0,1\,,\end{aligned} \tag{19}$$

$$\rho(h_i)(M)=\left(\sum_{j=-|i|}^{|i|+\theta(i)-1}M_{j,2|i|+\theta(i)}-\sum_{j=-|i|+1-\theta(i)}^{|i|-1}M_{j,2|i|+\theta(i)-1}+(\xi_1-\xi_0)\theta(-i)-\xi_1\right)(M), \quad i\in\mathbf{Z}, \tag{20}$$

$$\rho(c)(M)=(\xi_0-\xi_1)(M). \tag{21}$$

Above and throughout $[x]=\frac{q^x-q^{-x}}{q-q^{-1}}\in\mathbf{C}[[h]]$. If a pattern from the right hand side of (19) does not belong to $\Gamma(\{M\})$, i.e., it is not a C-pattern, then the corresponding term has to be deleted. (The coefficients in front of all such patterns are undefined, they contain zero multiples in the denominators. Therefore an equivalent statement is that all terms with zeros in the denominators have to be removed). With this convention all coefficients in front of the $C-$patterns in r.h.s of (18)-(21) are well defined as elements from $\mathbf{C}[[h]]$.

Proposition 1. *The endomorphisms $\{\rho(e_i),\rho(f_i),\rho(h_i),\rho(c)\}_{i\in\mathbf{Z}}$ satisfy Eqs. (1)-(3) with $\rho(e_i),\rho(f_i),\rho(h_i)$ and $\rho(c)$ substituted for $e_i,f_i,h_i,$ and c, respectively.*

Proof. The proof is based on a direct verification of the relations (1)-(3). This verification is lengthy and nontrivial. The most difficult to check are the last Cartan relations in (1), corresponding to $i = j$. In order to show they hold, one has to prove as an intermediate step that the following identities are fulfilled:

$$\sum_{s=0}^{1}\sum_{j=1-k}^{k-1}\sum_{l=-k}^{k-1}(-1)^s\frac{\prod_{i\neq l=-k}^{k-1}[L_{i,2k}-L_{j,2k-1}+s-1]\prod_{i=1-k}^{k-2}[L_{i,2k-2}-L_{j,2k-1}+s-1]}{\prod_{i\neq j=1-k}^{k-1}[L_{i,2k-1}-L_{j,2k-1}+s][L_{i,2k-1}-L_{j,2k-1}+s-1]}\times$$
$$\times\frac{\prod_{i=-k}^{k}[L_{i,2k+1}-L_{l,2k}+s]\prod_{i\neq j=1-k}^{k-1}[L_{i,2k-1}-L_{l,2k}+s]}{\prod_{i\neq l=-k}^{k-1}[L_{i,2k}-L_{l,2k}+s][L_{i,2k}-L_{l,2k}+s-1]}$$
$$=\left[\sum_{j=-k+1}^{k-1}L_{j,2k-1}-\sum_{j=-k+1}^{k-2}L_{j,2k-2}-\sum_{j=-k}^{k}L_{j,2k+1}+\sum_{j=-k}^{k-1}L_{j,2k}-1\right],\quad \forall k\in\mathbf{N},\quad (22)$$

$$\sum_{s=0}^{1}\sum_{j=-k}^{k-1}\sum_{l=-k}^{k}(-1)^s\frac{\prod_{i\neq l=-k}^{k}[L_{i,2k+1}-L_{j,2k}-s+1]\prod_{i=1-k}^{k-1}[L_{i,2k-1}-L_{j,2k}-s+1]}{\prod_{i\neq j=-k}^{k-1}[L_{i,2k}-L_{j,2k}-s][L_{i,2k}-L_{j,2k}-s+1]}\times$$
$$\times\frac{\prod_{i=-k-1}^{k}[L_{i,2k+2}-L_{l,2k+1}-s]\prod_{i\neq j=-k}^{k-1}[L_{i,2k}-L_{l,2k+1}-s]}{\prod_{i\neq l=-k}^{k}[L_{i,2k+1}-L_{l,2k+1}-s][L_{i,2k+1}-L_{l,2k+1}-s+1]}$$
$$=\left[\sum_{j=-k-1}^{k}L_{j,2k+2}-\sum_{j=-k}^{k}L_{j,2k+1}-\sum_{j=-k}^{k-1}L_{j,2k}+\sum_{j=-k+1}^{k-1}L_{j,2k-1}-1\right],\quad \forall k\in\mathbf{N}.\quad (23)$$

We proceed to verify (22) and (23). Let

$$F_k(h)=\sum_{s=0}^{1}\sum_{j=1-k}^{k-1}\sum_{l=-k}^{k-1}(-1)^s\frac{\prod_{i\neq l=-k}^{k-1}[L_{i,2k}-L_{j,2k-1}+s-1]\prod_{i=1-k}^{k-2}[L_{i,2k-2}-L_{j,2k-1}+s-1]}{\prod_{i\neq j=1-k}^{k-1}[L_{i,2k-1}-L_{j,2k-1}+s][L_{i,2k-1}-L_{j,2k-1}+s-1]}\times$$
$$\times\frac{\prod_{i=-k}^{k}[L_{i,2k+1}-L_{l,2k}+s]\prod_{i\neq j=1-k}^{k-1}[L_{i,2k-1}-L_{l,2k}+s]}{\prod_{i\neq l=-k}^{k-1}[L_{i,2k}-L_{l,2k}+s][L_{i,2k}-L_{l,2k}+s-1]},$$
$$G_k(h)=\left[\sum_{j=-k+1}^{k-1}L_{j,2k-1}-\sum_{j=-k+1}^{k-2}L_{j,2k-2}-\sum_{j=-k}^{k}L_{j,2k+1}+\sum_{j=-k}^{k-1}L_{j,2k}-1\right].$$

$F_k(h)$ and $G_k(h)$ are formal power series,

$$F_k(h)=\sum_{i=0}^{\infty}c_{ki}h^i,\quad G_k(h)=\sum_{i=0}^{\infty}d_{ki}h^i.$$

In order to prove that (22) holds one has to show that $c_{ki}=d_{ki}$ for any k and i. To this end replace h in (22) by a complex variable $x\in\mathbf{C}$, so that x takes values on any curve $\gamma\subset\mathbf{C}$ and $x\notin i\pi\mathbf{Q}$. Then $F_k(x)$ and $G_k(x)$ are well defined for any $x\in\gamma$ and

$$F_k(x)=\sum_{i=0}^{\infty}c_{ki}x^i,\quad G_k(x)=\sum_{i=0}^{\infty}d_{ki}x^i.$$

Therefore $c_{ki} = d_{ki}$ if $F_k(x) = G_k(x)$. Since h appears in (22) only through $q = e^{h/2}$, we have to show that (22) is an identity for q being a number, which is not a root of 1. The same arguments hold also for (23).

Let $q \in \mathbf{C}$ be not a root of 1. Setting $q^{2L_{i,2k-1}} = A_i$, $q^{2L_{i,2k}} = B_i$, $q^{2L_{i,2k+1}} = C_i$, $q^{2L_{i,2k-2}} = D_i$, $2k = n$ in (22) and $q^{2(L_{i,2k}-1)} = A_i$, $q^{2L_{i,2k+1}} = B_i$, $q^{2(L_{i,2k+2}-1)} = C_i$, $q^{2L_{i,2k-1}} = D_i$, $2k+1 = n$ in (23) and relabeling the indices in an appropriate way, one reduces both identities to the following one:

$$\sum_{j=1}^{n-1}\sum_{l=1}^{n} \frac{q\prod_{i\neq l=1}^{n}(A_j - q^{-2}B_i)\prod_{i=1}^{n-2}(A_j - q^{-2}D_i)\prod_{i=1}^{n+1}(B_l - C_i)\prod_{i\neq j=1}^{n-1}(B_l - A_i)}{A_jB_l\prod_{i\neq j=1}^{n-1}(A_j - A_i)(A_j - q^{-2}A_i)\prod_{i\neq l=1}^{n}(B_l - B_i)(B_l - q^{-2}B_i)}$$

$$-\sum_{j=1}^{n-1}\sum_{l=1}^{n} \frac{q^{-1}\prod_{i\neq l=1}^{n}(A_j - B_i)\prod_{i=1}^{n-2}(A_j - D_i)\prod_{i=1}^{n+1}(B_l - q^2C_i)\prod_{i\neq j=1}^{n-1}(B_l - q^2A_i)}{A_jB_l\prod_{i\neq j=1}^{n-1}(A_j - A_i)(A_j - q^2A_i)\prod_{i\neq l=1}^{n}(B_l - B_i)(B_l - q^2B_i)}$$

$$= (q - q^{-1})\left(1 - q^2\frac{\prod_{i=1}^{n-2}D_i\prod_{i=1}^{n+1}C_i}{\prod_{i=1}^{n-1}A_i\prod_{i=1}^{n}B_i}\right), \tag{24}$$

which can be written also in the form

$$\sum_{j=1}^{n-1}\frac{q\prod_{i=1}^{n}(A_j - q^{-2}B_i)\prod_{i=1}^{n-2}(A_j - q^{-2}D_i)}{A_j\prod_{i\neq j=1}^{n-1}(A_j - A_i)(A_j - q^{-2}A_i)}\sum_{l=1}^{n}\frac{\prod_{i=1}^{n+1}(B_l - C_i)\prod_{i\neq j=1}^{n-1}(B_l - A_i)}{(A_j - q^{-2}B_l)B_l\prod_{i\neq l=1}^{n}(B_l - B_i)(B_l - q^{-2}B_i)}$$

$$-\sum_{l=1}^{n}\frac{q^{-1}\prod_{i=1}^{n+1}(B_l - q^2C_i)\prod_{i=1}^{n-1}(B_l - q^2A_i)}{B_l\prod_{i\neq l=1}^{n}(B_l - B_i)(B_l - q^2B_i)}\sum_{j=1}^{n-1}\frac{\prod_{i\neq l=1}^{n}(A_j - B_i)\prod_{i=1}^{n-2}(A_j - D_i)}{A_j(B_l - q^2A_j)\prod_{i\neq j=1}^{n-1}(A_j - A_i)(A_j - q^2A_i)}$$

$$= (q - q^{-1})\left(1 - q^2\frac{\prod_{i=1}^{n-2}D_i\prod_{i=1}^{n+1}C_i}{\prod_{i=1}^{n-1}A_i\prod_{i=1}^{n}B_i}\right). \tag{25}$$

Consider the complex functions:

$$f_1(z) = \frac{\prod_{i=1}^{n+1}(z - C_i)\prod_{i\neq j=1}^{n-1}(z - A_i)}{(z - q^2A_j)\prod_{i=1}^{n}(z - B_i)(z - q^{-2}B_i)}, \quad f_2(z) = \frac{\prod_{i\neq l=1}^{n}(z - B_i)\prod_{i=1}^{n-2}(z - D_i)}{(z - q^{-2}B_l)\prod_{i=1}^{n-1}(z - A_i)(z - q^2A_i)}. \tag{26}$$

The function $f_1(z)$ (resp. $f_2(z)$) is holomorphic over $\mathbf{C}$ except in its simple poles q^2A_j, $B_1, \ldots, B_n$, $q^{-2}B_1, \ldots, q^{-2}B_n$ (resp. $q^{-2}B_l$, $A_1, \ldots, A_{n-1}$, $q^2A_1, \ldots, q^2A_{n-1}$.) Let C be a closed curve whose interior contains all poles of $f_1(z)$ (resp. $f_2(z)$.) The Residue theorem of complex analysis implies that $\oint_{\mathbf{C}} f_j(z)dz = 2\pi i\sum Res(f_j(z))$, $j = 1, 2$. On the other hand,

$$\oint_{\mathbf{C}} f_j(z)dz = -2\pi i Res f_j(\infty) \quad j = 1, 2. \tag{27}$$

Applying the Residue Theorem for both functions (26) and inserting the results in (25) one obtains:

$$\sum_{j=1}^{n-1}\frac{\prod_{i=1}^{n-2}(A_j - q^{-2}D_i)\prod_{i=1}^{n+1}(A_j - q^{-2}C_i)}{A_j\prod_{i\neq j=1}^{n-1}(A_j - A_i)\prod_{i=1}^{n}(A_j - q^{-4}B_i)} + \sum_{l=1}^{n}\frac{\prod_{i=1}^{n+1}(B_l - q^2C_i)\prod_{i=1}^{n-2}(B_l - q^2D_i)}{B_l\prod_{i\neq l=1}^{n}(B_l - B_i)\prod_{i=1}^{n-1}(B_l - q^4A_i)}$$

$$= 1 - q^2 \frac{\prod_{i=1}^{n-2} D_i \prod_{i=1}^{n+1} C_i}{\prod_{i=1}^{n-1} A_i \prod_{i=1}^{n} B_i}. \tag{28}$$

In order to prove this last identity consider the function:

$$f(z) = \frac{\prod_{i=1}^{n-2}(z - q^{-2}D_i) \prod_{i=1}^{n+1}(z - q^{-2}C_i)}{z \prod_{i=1}^{n-1}(z - A_i) \prod_{i=1}^{n}(z - q^{-4}B_i)}. \tag{29}$$

Applying again the Residue Theorem on this function one obtains (28) which proves formulas (22) and (23).

In addition to the identities (22) and (23) the last Cartan relation (1) is valid if:

$$\sum_{i=1}^{n} \left(\frac{\prod_{j=1}^{n-1}[a_i - b_j - 1] \prod_{j=1}^{n-1}[a_i - c_j - 1]}{\prod_{j\neq i=1}^{n}[a_i - a_j][a_i - a_j - 1]} - \frac{\prod_{j=1}^{n-1}[a_i - b_j] \prod_{j=1}^{n-1}[a_i - c_j]}{\prod_{j\neq i=1}^{n}[a_i - a_j][a_i - a_j + 1]} \right) = 0, \tag{30}$$

which is proved in a similar way. The verification of all other relations (1)-(3) is based on (30) or/and on some of the following identities:

$$\begin{aligned}
&([a-b-1][c-b-1] - [2][a-b][c-b-1] + [a-b][c-b]) \left(\frac{[a-d][c-e-1]}{[d-e-1][c-a-1]} + \frac{[c-d-1][a-e]}{[d-e+1][c-a-1]} \right) + \\
&([a-b-1][c-b-1] - [2][a-b-1][c-b] + [a-b][c-b]) \left(\frac{[a-e-1][c-d]}{[d-e-1][c-a+1]} + \frac{[a-d-1][c-e]}{[d-e+1][c-a+1]} \right) = 0,
\end{aligned} \tag{31}$$

$$\frac{[a-1][b-1] - [2][a][b-1] + [a][b]}{[a-b+1]} + \frac{[a-1][b-1] - [2][a-1][b] + [a][b]}{[a-b-1]} = 0, \tag{32}$$

$$[a-1] - [2][a] + [a+1] = 0. \tag{33}$$

This completes the proof. []

Remark. We may have constructed the endowmorphisms $\{\rho(e_i), \rho(f_i), \rho(h_i), \rho(c)\}_{i\in\mathbf{Z}}$ from the results on the representations of $U_h(gl_\infty)$, announced in Ref. [14] and, more precisely, from the endowmorphisms $\{\rho(E_i), \rho(F_i), \rho(H_i)\}_{i\in\mathbf{Z}}$ of the $\mathbf{C}[[h]]-$module $V(\{M\})$, which satisfy the Cartan and the Serre relations for $U_h(gl_\infty)$ (Ref. [14], Eqs. (16)-(18)). This possibility is based on the observation that the $C[[h]]-$linear map φ, defined on the generators as

$$\begin{aligned}
&\varphi(E_i) = e_i, \quad \varphi(F_i) = f_i, \quad \varphi(c) = c, \\
&\varphi(H_i) = h_i + (\theta(-i) + \alpha)c, \quad \alpha \in \mathbf{C}[h]
\end{aligned} \tag{34}$$

and extended by associativity is an (algebra) isomorphism of $U_h(gl_\infty) \oplus \mathbf{C}[[h]]c$ onto $U_h(a_\infty)$. Then the endomorphisms $\{\rho(e_i), \rho(f_i), \rho(h_i), \rho(c)\}_{i\in\mathbf{Z}}$ defined according to (34) (with $\alpha(\xi_0 - \xi_1) = \xi_1$) as

$$\begin{aligned}
&\rho(e_i) = \rho(E_i), \quad \rho(f_i) = \rho(F_i), \quad \rho(c) = \xi_0 - \xi_1, \\
&\rho(h_i) = \rho(H_i) - ((\xi_0 - \xi_1)\theta(-i) + \xi_1),
\end{aligned} \tag{35}$$

obey Eqs. (18)-(21). Here we have given a direct proof of *Proposition 1*, since the corresponding *Proposition 1* in Ref. [14] was only stated. Its proof would have been based again on the identities (22) and (23).

Consider ρ as a $\mathbf{C}[[h]]-$linear operator from $U_h(a_\infty)$ into $End\, V(\{M\};\xi_0,\xi_1)$. So far ρ is defined only on the Chevalley generators $\{e_i,f_i,h_i,c\}_{i\in\mathbf{Z}}$. Extend the domain of its definition on $\hat{U}_h(a_\infty)$: if ρ has already been defined on $a,b\in\hat{U}_h(a_\infty)$, then set

$$\rho(\alpha a+\beta b)=\alpha\rho(a)+\beta\rho(b),\quad \rho(ab)=\rho(a)\rho(b),\quad a,b\in\hat{U}_h(a_\infty),\quad \alpha,\beta\in\mathbf{C}[[h]]. \tag{36}$$

As we know from (10), any element $u\in U_h(a_\infty)$ can be represented as a sum $u=\sum_{i=0}^{\infty}u_ih^i,\ u_i\in\hat{U}_h(a_\infty)$. Then for an arbitrary $v\in V(\{M\};\xi_0,\xi_1)$, writing it as in (14), we have

$$\left(\sum_{i=0}^{\infty}\rho(u_i)h^i\right)v=\left(\sum_{i=0}^{\infty}\rho(u_i)h^i\right)\left(\sum_{j=0}^{\infty}v_jh^j\right)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\rho(u_{n-m})v_m\right)h^n\in V(\{M\};\xi_0,\xi_1), \tag{37}$$

since

$$\sum_{m=0}^{n}\rho(u_{n-m})v_m\in\hat{V}(\{M\};\xi_0,\xi_1).$$

Using (37), we extend ρ on $U_h(a_\infty)$:

$$\rho(u)=\sum_{i=0}^{\infty}\rho(u_i)h^i\in End\, V(\{M\};\xi_0,\xi_1)\quad\forall\ u\in U_h(a_\infty). \tag{38}$$

Hence ρ is a well defined map from $U_h(a_\infty)$ into $End\, V(\{M\};\xi_0,\xi_1)$,

$$\rho:\ U_h(a_\infty)\to End\, V(\{M\};\xi_0,\xi_1). \tag{39}$$

Proposition 2. *The map (39), acting on the $C-$basis according to Eqs. (18)-(21), defines a highest weight irreducible representation of $U_h(a_\infty)$ in $V(\{M\};\xi_0,\xi_1)$).*

Proof. According to *Proposition 1*, (36), (38), ρ is a $\mathbf{C}[[h]]$-homomorphism of $U_h(a_\infty)$ in $End\, V(\{M\};\xi_0,\xi_1)$. It is continuous in the $h-$adic topology. Indeed, let $u\in U_h(a_\infty)$. Then any neighbourhood $W(\rho(u))$ of $\rho(u)$ contains a basic neighbourhood $\rho(u)+h^nEnd\, V(\{M\};\xi_0,\xi_1)\subset W(\rho(u))$. Evidently

$$\rho\left(u+h^nU_h(a_\infty)\right)\subset\rho(u)+h^nEnd\, V(\{M\};\xi_0,\xi_1)\subset W(\rho(u)) \tag{40}$$

and therefore ρ is continuous in u for any $u\in U_h(a_\infty)$. Hence $V(\{M\};\xi_0,\xi_1)$ is a $U_h(a_\infty)-$module. It is a highest weight module with respect to the "Borel" subalgebra $U_h(n_+)$. The highest weight vector $(\hat{M})$, which by definition satisfies the condition $\rho(U_h(n_+))(\hat{M})=0$ and is an eigenvector of $\rho(H)$, corresponds to the one from (11) with

$$\hat{M}_{i,2k+\theta-1}=M_i,\quad\forall\, k\in\mathbf{N},\quad\theta=0,1,\qquad i\in[-\theta-k+1,k-1]. \tag{41}$$

Moreover, $V(\{M\};\xi_0,\xi_1)=\rho(U_h(a_\infty))(\hat{M})$. Since $V(\{M\};\xi_0,\xi_1)$ contains no other singular vectors, vectors annihilated from $\rho(U_h(n_+))$, each $V(\{M\};\xi_0,\xi_1)$ is an irreducible $U_h(a_\infty)$-module. The proof of the latter follows from the results in Ref. [12] and the observation that each (deformed) matrix element in the transformation relations (18)-(21) is zero only if the corresponding nondeformed matrix element vanishes. []

3. REPRESENTATIONS OF THE ALGEBRA $\mathbf{U}_h(A_\infty)$

In this section we show that some of the $U_h(a_\infty)-$modules $V(\{M\};\xi_0,\xi_1)$ can be turned into irreducible highest weight $U_h(A_\infty)-$modules. First, following Ref. [8], we recall the definition of the quantum algebra $U_h(A_\infty)$ (denoted by the authors as $U_h(g'(A_\infty))$) only within the algebra sector (for the other Hopf algebra maps see Ref. [8]). $U_h(A_\infty)$ consists of all elements

$$u=\sum_{k=0}^{\infty}h^k\sum_{l=0}^{l(k)}c^l\sum_{\alpha,\beta\in Q'_+}\ \sum_{\gamma(k,l)\in\mathbf{Z}_+^\infty}\ \sum_{t=1}^{t(\alpha,\beta)}\mathcal{F}_{\alpha,k,l,t}\prod_{i\in\mathbf{Z}}h_i^{\gamma(k,l)_i}\mathcal{E}_{\beta,k,l,t},\ \ \textit{infinite sums over } \alpha,\ \beta,\ \gamma, \tag{42}$$

however certain conditions on the pairs (α,γ) corresponding to the non-zero summands are imposed. In order to state them, set for $\alpha=\sum_i m_i\alpha_i\in\mathbf{Q}'_+,\ \gamma=\{\gamma_i\}_{i\in\mathbf{Z}}\in\mathbf{Z}_+^\infty$,

$$S(\alpha)=\{i\mid m_i\neq 0\},\quad S(\gamma)=\{i\mid \gamma_i\neq 0\},\quad S(\alpha,\gamma)=S(\alpha)\cup S(\gamma). \tag{43}$$

Connecting i and j if $|i-j|=1$, one can view $S(\alpha,\gamma)$ as a graph. Denote by $\mathcal{F}(\alpha,\gamma)$ the collection of its connected components. For any $u=\sum_{k=0}^\infty h^k u_k$ as given in (42) consider the series

$$u_k=\sum_{l=0}^{l(k)}c^l\sum_{\alpha,\beta\in Q'_+}\ \sum_{\gamma(k,l)\in\mathbf{Z}_+^\infty}\ \sum_{t=1}^{t(\alpha,\beta)}\mathcal{F}_{\alpha,k,l,t}\prod_{i\in\mathbf{Z}}h_i^{\gamma(k,l)_i}\mathcal{E}_{\beta,k,l,t}, \tag{44}$$

and let

$$\mathcal{F}(u,k)=\cup\mathcal{F}(\alpha,\gamma), \tag{45}$$

where the union is taken over all α and γ, which appear in the non-zero summands of (44). For $i\in\mathbf{Z}$ and $k\in\mathbf{Z}_+$ set

$$Int(u,k,i)=\{I\in\mathcal{F}(u,k)|\ i\in I\}. \tag{46}$$

For $r\in\mathbf{N}$ define the series $u(r)$, corresponding to u, by substituting 0 for all h_i ($i\leq -r$ or $i>r$) and for all $e_i,\ f_i$ ($|i|\geq r$).

Definition 1. [8] *The series u of the form (42) is said to belong to $U_h(A_\infty)$, provided*

(i) for any $k\in\mathbf{Z}_+$ and any $i\in\mathbf{Z}$ the set $Int(u,k,i)$ is finite;

(ii) $u(r)\in U_h(a_\infty)$ for all $r\in\mathbf{N}$.

We turn to construct a class of representations of $U_h(A_\infty)$. The idea is to show that within certain $V(\{M\};\xi_0,\xi_1)$ the domain of the definition of the operator $\rho:\ U_h(a_\infty)\to End\ V(\{M\};\xi_0,\xi_1)$ (see (39)) can be extended from $U_h(a_\infty)$ to $U_h(A_\infty)$, so that ρ is a representation of $U_h(A_\infty)$ in $V(\{M\};\xi_0,\xi_1)$. The construction is a natural one. We assume that $\rho(u)$ is a continuous $\mathbf{C}[[h]]-$linear operator in $V(\{M\};\xi_0,\xi_1)$ for any $u\in U_h(A_\infty)$. Then for any $v=\sum_{j=0}^\infty v_j h^j\in V(\{M\};\xi_0,\xi_1),\ v_j\in\hat{V}(\{M\};\xi_0,\xi_1)$

$$\rho(u)v=\sum_{j=0}^{\infty}\left(\rho(u)v_j\right)h^j. \tag{47}$$

The above series is well defined, if $\rho(u)v_j\in V(\{M\};\xi_0,\xi_1)$. Since v_j is a (finite) $\mathbf{C}[[h]]-$linear combination of $C-$basis vectors the latter holds, and hence $\rho(u)$ is defined as an operator in $V(\{M\};\xi_0,\xi_1)$, if $\rho(u)(M)\in$

$V(\{M\};\xi_0,\xi_1)$ for any $C-$pattern (M). Thus, the first step is to clarify which are the $U_h(a_\infty)-$modules $V(\{M\};\xi_0,\xi_1)$, for which this can be achieved.

Definition 2. *We say that $V(\{M\};\xi_0,\xi_1)$ is of a finite-signature or, more precisely, of (m,n) signature and write $\{M\}=\{M\}_{m,n}$ if there exist integers $m\le n\in\mathbf{Z}$, such that $M_m=M_{m-k}$ and $M_n=M_{n+k}$ for all $k\in\mathbf{N}$.*

We now proceed to show that each finite signature $U_h(a_\infty)$ module $V(\{M\}_{m,n};M_m,M_n)$ can be turned into a $U_h(A_\infty)$ module. To this end we prove first a few preliminary propositions.

Denote by $V(\{M\};\xi_0,\xi_1)_N,\ 1<N\in\mathbf{N}$ the subspace of $V(\{M\};\xi_0,\xi_1)$, which is a $\mathbf{C}[[h]]-$linear envelope of all $C-$basis vectors (M), for which

$$M_{i,2k+\theta-1}=M_i\ \ \forall\, 2k+\theta-1\ge N,\quad \theta=0,1,\ \ i\in[1-\theta-k,k-1]; \tag{48}$$

Note that $V(\{M\};\xi_0,\xi_1)_N$ is a finite dimensional subspace.

Proposition 3.

$$\rho(e_k)V(\{M\};\xi_0,\xi_1)_N=0,\ \ if\ \ k\notin(-\frac{1}{2}(N+1),\frac{1}{2}(N-2)). \tag{49}$$

Proof. From (19) one obtains:

$$\begin{aligned}\rho(e_k)(M)=&-\sum_{j=-k}^{k}\ \sum_{l=-k-1}^{k}S(j,l;0)\\ &\times\left|\frac{\prod_{i\ne l=-k-1}^{k}[L_{i,2k+2}-L_{j,2k+1}-1]\prod_{i=-k}^{k-1}[L_{i,2k}-L_{j,2k+1}-1]}{\prod_{i\ne j=-k}^{k}[L_{i,2k+1}-L_{j,2k+1}][L_{i,2k+1}-L_{j,2k+1}-1]}\right|^{1/2}\\ &\times\left|\frac{\prod_{i=-k-1}^{k+1}[L_{i,2k+3}-L_{l,2k+2}]\prod_{i\ne j=-k}^{k}[L_{i,2k+1}-L_{l,2k+2}]}{\prod_{i\ne l=-k-1}^{k}[L_{i,2k+2}-L_{l,2k+2}][L_{i,2k+2}-L_{l,2k+2}-1]}\right|^{1/2}(M)^{\{l,2k+2\}}_{\{j,2k+1\}},\ k\ge 0.\end{aligned} \tag{50}$$

$$\begin{aligned}\rho(e_{-k})(M)=&-\sum_{j=-k+1}^{k-2}\ \sum_{l=-k+1}^{k-1}S(j,l;1)\\ &\times\left|\frac{\prod_{i\ne l=-k+1}^{k-1}[L_{i,2k-1}-L_{j,2k-2}+1]\prod_{i=2-k}^{k-2}[L_{i,2k-3}-L_{j,2k-2}+1]}{\prod_{i\ne j=-k+1}^{k-2}[L_{i,2k-2}-L_{j,2k-2}][L_{i,2k-2}-L_{j,2k-2}+1]}\right|^{1/2}\\ &\times\left|\frac{\prod_{i=-k}^{k-1}[L_{i,2k}-L_{l,2k-1}]\prod_{i\ne j=-k+1}^{k-2}[L_{i,2k-2}-L_{l,2k-1}]}{\prod_{i\ne l=-k+1}^{k-1}[L_{i,2k-1}-L_{l,2k-1}][L_{i,2k-1}-L_{l,2k-1}+1]}\right|^{1/2}(M)^{-\{l,2k-1\}}_{-\{j,2k-2\}},\ k>1\end{aligned} \tag{51}$$

If $(M)\in V(\{M\};\xi_0,\xi_1)_N$ and

$$\begin{aligned}&if\quad k\ge\frac{1}{2}(N-2)),\ \ then\ \ L_{i,2k+3}=L_{i,2k+2}=L_i=M_i-i,\\ &if\quad k\le-\frac{1}{2}(N+1),\ \ then\ \ L_{i,2k-1}=L_{i,2k}=L_i=M_i-i.\end{aligned} \tag{52}$$

In both cases the r.h.s. of (50) and (51) contain zero multiples $(L_l - L_l)$ and therefore vanish. []

Proposition 4.

$$\rho(f_k)V(\{M\}_{m,n}; M_m, M_n)_N = 0, \ \ if \ \ k \notin (min\{-\frac{1}{2}(N+3), m-1\}, max\{\frac{1}{2}N, n\}). \tag{53}$$

Proof. Let $(M) \in V(\{M\}_{m,n}; M_m, M_n)_N$. The relations that follow from (19) in this case are

$$\begin{aligned}\rho(f_k)(M) = &-\sum_{j=-k}^{k}\sum_{l=-k-1}^{k} S(j,l;0)\\ &\times \left|\frac{\prod_{i\neq l=-k-1}^{k}[L_{i,2k+2}-L_{j,2k+1}]\prod_{i=-k}^{k-1}[L_{i,2k}-L_{j,2k+1}]}{\prod_{i\neq j=-k}^{k}[L_{i,2k+1}-L_{j,2k+1}][L_{i,2k+1}-L_{j,2k+1}+1]}\right|^{1/2}\\ &\times \left|\frac{\prod_{i=-k-1}^{k+1}[L_{i,2k+3}-L_{l,2k+2}+1]\prod_{i\neq j=-k}^{k}[L_{i,2k+1}-L_{l,2k+2}+1]}{\prod_{i\neq l=-k-1}^{k}[L_{i,2k+2}-L_{l,2k+2}][L_{i,2k+2}-L_{l,2k+2}+1]}\right|^{1/2} (M)^{-\{l,2k+2\}}_{-\{j,2k+1\}}, \ k\in \mathbf{Z}_+,\end{aligned} \tag{54}$$

$$\begin{aligned}\rho(f_{-k})(M) = &-\sum_{j=-k+1}^{k-2}\sum_{l=-k+1}^{k-1} S(j,l;1)\\ &\times \left|\frac{\prod_{i\neq l=-k+1}^{k-1}[L_{i,2k-1}-L_{j,2k-2}]\prod_{i=2-k}^{k-2}[L_{i,2k-3}-L_{j,2k-2}]}{\prod_{i\neq j=-k+1}^{k-2}[L_{i,2k-2}-L_{j,2k-2}][L_{i,2k-2}-L_{j,2k-2}-1]}\right|^{1/2}\\ &\times \left|\frac{\prod_{i=-k}^{k-1}[L_{i,2k}-L_{l,2k-1}-1]\prod_{i\neq j=-k+1}^{k-2}[L_{i,2k-2}-L_{l,2k-1}-1]}{\prod_{i\neq l=-k+1}^{k-1}[L_{i,2k-1}-L_{l,2k-1}][L_{i,2k-1}-L_{l,2k-1}-1]}\right|^{1/2} (M)^{\{l,2k-1\}}_{\{j,2k-2\}}, \ k>1.\end{aligned} \tag{55}$$

If $k \geq N/2$, then $L_{i,2k+3} = L_{i,2k+2} = L_{i,2k+1} = L_{i,2k} = L_i = M_i - i$ and therefore (54) reads:

$$\rho(f_k)(M) = -\sum_{j=-k}^{k}\sum_{l=-k-1}^{k} S(j,l;0)\left|\frac{\prod_{i\neq l=-k-1}^{k}[L_i-L_j]\prod_{i=-k}^{k-1}[L_i-L_j]}{\prod_{i\neq j=-k}^{k}[L_i-L_j][L_i-L_j+1]}\right|^{1/2}$$

$$\times \left|\frac{\prod_{i=-k-1}^{k+1}[L_i-L_l+1]\prod_{i\neq j=-k}^{k}[L_i-L_l+1]}{\prod_{i\neq l=-k-1}^{k}[L_i-L_l][L_i-L_l+1]}\right|^{1/2} (M)^{-\{l,2k+2\}}_{-\{j,2k+1\}}$$

Above only the term with $j = k$ survives:

$$\rho(f_k)(M) = -\sum_{l=-k-1}^{k} S(k,l;0)\left|\frac{\prod_{i\neq l=-k-1}^{k}[L_i-L_k]\prod_{i=-k}^{k-1}[L_i-L_k]}{\prod_{i=-k}^{k-1}[L_i-L_k][L_i-L_k+1]}\right|^{1/2}$$

$$\times\left|\frac{\prod_{i=-k-1}^{k+1}[L_i-L_l+1]\prod_{i=-k}^{k-1}[L_i-L_l+1]}{\prod_{i\neq l=-k-1}^{k}[L_i-L_l][L_i-L_l+1]}\right|^{1/2}(M)^{-\{l,2k+2\}}_{-\{k,2k+1\}}$$

$$= -S(k,k;0)\left|\frac{\prod_{i=-k-1}^{k-1}[L_i-L_k]\prod_{i=-k-1}^{k+1}[L_i-L_k+1]\prod_{i=-k}^{k-1}[L_i-L_k+1]}{\prod_{i=-k}^{k-1}[L_i-L_k+1]\prod_{i=-k-1}^{k-1}[L_i-L_k][L_i-L_k+1]}\right|^{1/2}(M)^{-\{k,2k+2\}}_{-\{k,2k+1\}}$$

$$-\sum_{l=-k-1}^{k-1} S(k,l;0)\left|\frac{\prod_{i\neq l=-k-1}^{k}[L_i-L_k]\prod_{i=-k-1}^{k+1}[L_i-L_l+1]\prod_{i=-k}^{k-1}[L_i-L_l+1]}{\prod_{i=-k}^{k-1}[L_i-L_k+1]\prod_{i\neq l=-k-1}^{k}[L_i-L_l][L_i-L_l+1]}\right|^{1/2}(M)^{-\{l,2k+2\}}_{-\{k,2k+1\}}.$$

In the last term $l \neq k$. Hence

$$\prod_{i\neq l=-k-1}^{k}[L_i-L_k] = [L_k-L_k]\prod_{i\neq l=-k-1}^{k-1}[L_i-L_k] = 0$$

and therefore it vanishes. Then

$$\rho(f_k)(M) = -\left|\frac{\prod_{i=-k-1}^{k+1}[L_i-L_k+1]}{\prod_{i=-k-1}^{k-1}[L_i-L_k+1]}\right|^{1/2}(M)^{-\{k,2k+2\}}_{-\{k,2k+1\}} = -\left|[L_{k+1}-L_k+1]\right|^{1/2}(M)^{-\{k,2k+2\}}_{-\{k,2k+1\}},$$

i.e.,

$$\rho(f_k)(M) = -\left|[M_{k+1}-M_k]\right|^{1/2}(M)^{-\{k,2k+2\}}_{-\{k,2k+1\}}. \qquad (56)$$

If $k \geq n$ then $M_{k+1} = M_k$ and $\rho(f_k)(M) = 0$. In a similar way one derives from (55) that $\rho(f_k)(M) = 0$ if $k \leq min\{-\frac{1}{2}(N+3), m-1\}$, which proves (53). []

Proposition 5.

$$\rho(h_k)V(\{M\}_{m,n}; M_m, M_n)_N = 0, \;\; if \;\; k \notin (min\{-\frac{1}{2}(N+1), m\}, max\{\frac{1}{2}N, n\}). \qquad (57)$$

The proof follows easily from (20).

Set

$$r_N = max\{\frac{1}{2}(N+3), 1-m, n\}. \qquad (58)$$

From the last three propositions one concludes.

Corollary 1. *If* $k \notin (-r_N, r_N)$, *then*

$$\rho(h_k)V(\{M\}_{m,n}; M_m, M_n)_N = \rho(e_k)V(\{M\}_{m,n}; M_m, M_n)_N = \rho(f_k)V(\{M\}_{m,n}; M_m, M_n)_N = 0. \qquad (59)$$

Proposition 6. $\rho(U_h(n_+)_\beta)V(\{M\};\xi_0,\xi_1)_N \subset V(\{M\};\xi_0,\xi_1)_N$ *for any* $N \in \mathbf{N}$. *More precisely,*

$$If \;\; S(\beta) \subset (-\frac{(N+1)}{2}, \frac{(N-2)}{2}) \equiv I_N, \;\; then \;\; \rho(U_h(n_+)_{+\beta})V(\{M\};\xi_0,\xi_1)_N \subset V(\{M\};\xi_0,\xi_1)_N. \quad (60)$$

$$If \;\; S(\beta) \not\subset \; I_N, \;\; then \;\; \rho(U_h(n_+)_\beta)V(\{M\};\xi_0,\xi_1)_N = 0. \qquad (61)$$

Proof. From (19) (or directly from (50) and (51)) one concludes that

$$\rho(e_j)(M) \in V(\{M\};\xi_0,\xi_1)_N, \ \ \forall \ j \in (-\frac{(N+1)}{2}, \frac{(N-2)}{2}) \ \ and \ \forall \ \ (M) \in V(\{M\};\xi_0,\xi_1)_N. \tag{62}$$

Hence (60) holds.

Assume $S(\beta) \not\subset \ I_N$ and let P_β be a monomial of the generators $e_i,\ i \in S(\beta)$. P_β can be represented as $P_\beta = Q' e_n Q$, where Q depends only on the generators e_j with $j \in S(\beta) \cap I_N$ and $n \notin I_N$. Then (62) yields that $\rho(Q)(M) \in V(\{M\};\xi_0,\xi_1)_N$ and therefore (*Proposition 3*) $\rho(P_\beta)(M) = \rho(Q')\rho(e_n)\rho(Q)(M) = 0$. []

Proposition 7. *Let u be any element from $U_h(A_\infty)$, represented as in (42). Then*

$$\sum_{k=0}^{\infty} h^k \sum_{l=0}^{l(k)} \rho(c)^l \sum_{\alpha,\beta \in Q'_+} \sum_{\gamma(k,l)\in \mathbf{Z}_+^\infty} \sum_{t=1}^{t(\alpha,\beta)} \rho(\mathcal{F}_{\alpha,k,l,t}) \prod_{i\in\mathbf{Z}} \rho(h_i)^{\gamma(k,l)_i} \rho(\mathcal{E}_{\beta,k,l,t})(M) \in V(\{M\}_{m,n}; M_m, M_n) \tag{63}$$

for any (M) from the basis $\Gamma(\{M\})$ of $V(\{M\}_{m,n}; M_m, M_n)$. For a fixed k the number of the non-zero summands in (63) is finite.

Proof. In order to prove the proposition it suffices to show that

$$\sum_{\alpha,\beta \in Q'_+} \sum_{\gamma(k,l)\in \mathbf{Z}_+^\infty} \sum_{t=1}^{t(\alpha,\beta)} \rho(\mathcal{F}_{\alpha,k,l,t}) \prod_{i\in\mathbf{Z}} \rho(h_i)^{\gamma(k,l)_i} \rho(\mathcal{E}_{\beta,k,l,t})(M) \in \hat{V}(\{M\}_{m,n}; M_m, M_n). \tag{64}$$

Let for definiteness $(M) \in V(\{M\}_{m,n}; M_m, M_n)_N$ and let β_0 be the weight of (M). Since $\mathcal{E}_{\beta,k,l,t} \in U_h(n_+)_\beta$, $\rho(\mathcal{E}_{\beta,k,l,t})(M) \in V(\{M\}_{m,n}; M_m, M_n)_N$ (*Proposition 6*). Each $\rho(\mathcal{E}_{\beta,k,l,t})(M)$ has a weight $\beta + \beta_0$ and the nonzero vectors $\rho(\mathcal{E}_{\beta,k,l,t})(M)$, corresponding to different $\beta \in \mathbf{Q'_+}$, are linearly independent (over $\mathbf{C}[[h]]$). Since $V(\{M\}_{m,n}; M_m, M_n)_N$ is a finite dimensional subspace, $\rho(\mathcal{E}_{\beta,k,l,t})(M) \neq 0$ only for a finite number of $\beta \in \mathbf{Q'_+}$. Hence the sum over β in (64) is finite. Setting $\rho(\mathcal{E}_{\beta,k,l,t})(M) = v_{\beta,k,l,t} \in \hat{V}(\{M\}_{m,n}; M_m, M_n)_N$, we obtain for the l.h.s. of (64)

$$\sum_{\alpha,\beta \in Q'_+} \sum_{\gamma(k,l)\in \mathbf{Z}_+^\infty} \sum_{t=1}^{t(\alpha,\beta)} \rho(\mathcal{F}_{\alpha,k,l,t}) \prod_{i\in\mathbf{Z}} \rho(h_i)^{\gamma(k,l)_i} v_{\beta,k,l,t}. \tag{65}$$

We proceed to show that the sum over α and γ in (65) is finite too. Without loss of generality we assume that every $\mathcal{F}_{\alpha,k,l,t} \prod_{i\in\mathbf{Z}} h_i^{\gamma(k,l)_i}$ is a monomial of $\{f_i, h_i\}_{i\in\mathbf{Z}}$. Consider any term from (65), corresponding to a particular pair (α,γ):

$$\rho(\mathcal{F}_{\alpha,k,l,t}) \prod_{i\in\mathbf{Z}_+} \rho(h_i)^{\gamma(k,l)_i} v_{\beta,k,l,t}. \tag{66}$$

Let $\mathcal{F}(\alpha,\gamma)$ be the connected components of $S(\alpha,\gamma)$:

$$\mathcal{F}(\alpha,\gamma) = \{I_{a_i,b_i} \equiv [a_i,b_i] \mid i = 1,2,\ldots,n\}, \quad a_i \le b_i \in \mathbf{Z}, \ \ |a_i - b_j| > 1, \ if \ i \neq j. \tag{67}$$

In (67) we do not distinguish between a connected component I_{a_i,b_i} (with a beginning in a_i and end in b_i) and the corresponding to it finite integer interval $[a_i,b_i]$. Then

$$\mathcal{F}_{\alpha,k,l,t} \prod_{i\in\mathbf{Z}_+} (h_i)^{\gamma(k,l)_i} = \prod_{i=1}^{n} \mathcal{F}_i, \tag{68}$$

where $\mathcal{F}_i$ is a monomial of $f_j, h_j,\ j \in [a_i, b_i]$. From (1) and (3) it follows that the multiples $\mathcal{F}_i$ in (68) commute. If $[a_i, b_i] \subset (-\infty, -r_N]$ or $[a_i, b_i] \subset [r_N, \infty)$, then *Corollary 1* yields that $\rho(\mathcal{F}_i)v_{\beta,k,l,t}$=0. Hence also $\rho(\mathcal{F}_{\alpha,k,l,t}) \prod_{i\in\mathbf{Z}_+} \rho(h_i)^{\gamma(k,l)_i} v_{\beta,k,l,t} = 0$. Thus, the sum in (64) is over such pairs (α, γ), for which all connected components of $S(\alpha, \gamma)$, namely the elements from $\mathcal{F}(\alpha, \gamma)$, have nonzero intersection with $(-r_N, r_N)$. There is only a finite number of pairs (α, γ) with this property, for which $Int(u, k, i)$ is finite.

The conclusion is that the l.h.s. of the series (64) contains a finite number of non-zero summands. Since the generators of $U_h(A_\infty)$ (see (18)-(21)) transform $\hat{V}(\{M\}_{m,n}; M_m, M_n)$ into itself, (64) holds. Hence (63) holds too. []

Based on *Proposition 7*, we extend the domain of the operator ρ on $U_h(A_\infty)$, setting for any $u \in U_h(A_\infty)$ (see (42))

$$\rho(u) = \sum_{k=0}^{\infty} h^k \sum_{l=0}^{l(k)} \rho(c)^l \sum_{\alpha,\beta\in Q'_+} \sum_{\gamma(k,l)\in\mathbf{Z}_+^\infty} \sum_{t=1}^{t(\alpha,\beta)} \rho(\mathcal{F}_{\alpha,k,l,t}) \prod_{i\in\mathbf{Z}} \rho(h_i)^{\gamma(k,l)_i} \rho(\mathcal{E}_{\beta,k,l,t}) \in End V(\{M\}_{m,n}; M_m, M_n) \tag{69}$$

The map ρ is a homomorphism of $U_h(A_\infty)$ in $End\ V(\{M\}_{m,n}; M_m, M_n)$. It is continuous in the $h-$adic topology. Hence ρ defines a representation of $U_h(A_\infty)$ in $V(\{M\}_{m,n}; M_m, M_n)$, which is a highest weight irreducible representation (since it is a highest weight irreps with respect to the subalgebra $U_h(a_\infty)$).

Let $v = \sum_{k\in\mathbf{Z}_+} h^k v_k,\ \ v_k \in \hat{V}(\{M\}_{m,n}; M_m, M_n)$ be an arbitrary element from $V(\{M\}_{m,n}; M_m, M_n)$. Since each v_k is a finite linear combination of C-vectors, for any $k \in \mathbf{Z}_+$ there exists an integer $N_k > 1$ such that $v_k \in V(\{M\}_{m,n}; M_m, M_n)_{N_k}$. Then from (59) and (61) one concludes that the following properties hold:

$$1.\quad \rho(U_h(n_+)_\beta)v_k = 0,\ \ if\ \ S(\beta) \not\subset (-r_{N_k}, r_{N_k}), \tag{70a}$$

$$2.\quad \rho(U_h(n_-)_{-\alpha})v_k = 0,\ \ if\ \ S(\alpha) \subset (-\infty, -r_{N_k}]\ \ or\ \ S(\alpha) \subset [r_{N_k}, \infty), \tag{70b}$$

$$3.\quad \rho(h_i)v_k = 0,\ \ if\ \ |i| \geq r_{N_k}. \tag{70c}$$

Therefore each finite signature representation of $U_h(A_\infty)$ in $V(\{M\}_{m,n}; M_m, M_n)$ is a restricted representation (see *Definition 4.1* in Ref. [8]).

Let us mention in conclusion that the $U_h(a_\infty)$ modules $V(\{M\}; \xi_0, \xi_1)$, which are not of a finite (m, n) signature and for which $\xi_0 \neq M_m,\ \xi_1 \neq M_n$, cannot be turned into $U_h(A_\infty)$ modules. In order to see this consider the transformation of the highest weight vector $(\hat{M})$ under the action of the operator $I = \sum_{i\in\mathbf{Z}} h_i \in U_h(A_\infty)$. From (20) and (41) one obtains:

$$\rho(I)(\hat{M}) = \left(\sum_{i=-\infty}^{0} (M_i - \xi_0) + \sum_{i=1}^{\infty} (M_i - \xi_1) \right) (\hat{M}). \tag{71}$$

The r.h.s. of (71) is not divergent only in the finite-signature modules $V(\{M\}_{m,n}; M_m, M_n)$.

We were dealing with algebras and modules over $\mathbf{C}[[h]]$. The algebra $U_h(A_\infty)$ is however well defined also for h being a fixed complex number h_c, such that $h_c \notin i\pi\mathbf{Q}$, namely in the case $q = e^{h_c}$ is not a root of 1 [8]. In that case the representations we have obtained remain highest weight irreps of $U_{h_c}(A_\infty)$ realized in infinite-dimensional complex linear spaces.

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References

1. Date, E., Jimbo, M., Kashiwara M., Miwa, T.: Operator Approach to the Kadomtsev-Petviashvili Equation - Transformation Groups for Soliton Equations III. J. Phys. Soc. Japan **50**, 3806-3818 (1981).
2. Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for soliton equations - Euclidean Lie algebras and reduction of the KP hierarchy. Publ. RIMS Kyoto Univ. **18**, 1077-1110 (1982).
3. Drinfeld, V.: *Quantum groups.* ICM proceedings, Berkeley 798-820 (1986).
4. Goddard, P., Olive,D.: Kac-Moody and Virasoro Algebras in Relation to Quantum Physics. Int. J. Mod. Phys. **A 1**, 303-414 (1986).
5. Haldane, F.D.M.: "Fractional statistics" in arbitrary dimensions: a generalization of the Pauli principle. Phys. Rev. Lett. **67**, 937-940 (1991).
6. Kac, V.G., Peterson, D.H.: Spin and wedge representations of infinite-dimensional Lie algebras and groups. Proc. Natl. Acad. Sci. USA **78**, 3308-3312 (1981).
7. Karabali, D., Nair,V.P.: Many-body states and operator algebra for exclusion statistics. Nucl. Phys. **B 438**, 551-560 (1995).
8. Levendorskii, S., Soibelman, Y.: Quantum group A_∞. Commun. Math. Phys. **140**, 399-414 (1991).
9. Palev, T.D.: Lie algebraical aspects of the quantum statistics. Thesis, Institute of Nuclear Research and Nuclear Energy (1976), Sofia; Palev, T.D.: Lie algebraic aspects of quantum statistics. Unitary quantization (A-quantization). Preprint JINR E17-10550 (1977) and hep-th/9705032.
10. Palev, T.D.: Lie superalgebras, infinite-dimensional algebras and quantum statistics. Rep. Math. Phys. **31**, 241-262 (1992).
11. Palev, T.D.: Representations with a highest weight of the Lie algebra a_∞. Funkt. Anal. Prilozh. **24**, No 3, 88-89 (1990) (in Russian); Funct. Anal. Appl. **24**, 250-251 (1990) (English translation).
12. Palev, T.D.: Highest weight irreducible unitarizable representations of the Lie algebras of infinite matrices. I. The algebra A_∞. Journ. Math. Phys. **31**, 1078-1084 (1990).
13. Palev, T.D., Stoilova, N.I.: Many-body Wigner quantum systems. Journ. Math. Phys. **38**, 2506-2523 (1997) and hep-th/9606011.
14. Palev, T.D., Stoilova, N.I.: Highest weight representations of the quantum algebra $U_h(gl_\infty)$. Preprint ICTP IC/97/29 (1997) and q-alg/9704001.